

## Symbolically Solving Differential Equations

## Objectives:

1. To learn about the MATLAB Symbolic Solver
2. To expand knowledge of solutions to Diff-EQs

## 1 Symbolically Solving Differential Equations

The Symbolic Toolbox provides a link between MATLAB and the symbolic algebra program know as Maple. Unlike the rest of MATLAB, which provides powerful numerical routines, the Symbolic Toolbox deals with symbols, formulas, and equations. We will use the Symbolic Toolbox to solve differential equations – which means that MATLAB will try to tell you the exact, analytic solution to an equations, or even to an initial value problem. MATLAB can be successful, only if such a solution exists!

Whenever you want to use symbolic mathematics, you must use the `syms` operator to alert MATLAB that you are using a symbolic variable, rather than giving it a specific value. Enter

```
>> g=sym('x^2')
>> diff(g)
```

MATLAB recognizes that the input argument has class `sym` and promptly takes the derivative of  $g = x^2$  with respect to  $x$ . **Type `help diff`**. Notice that the help page gives no hint of the fact that the function `diff` takes the derivative of an object of class `sym` until the very last few lines. That's because the function `diff` can work with vectors of class `double` and `sym`. This technique is known as *overloading* the function. To indicate that you want to see how `diff` works on these objects, you have to type the command `help sym/diff`. Try it out!

Type the following commands, to learn what they do:

```
>> syms x a b c
>> f = z^2*x^2 + b*x + c
>> diff(f)
```

We can just as easily take a second derivative.

```
>> diff(f,2)
```

You may be wondering – how the Symbolic Toolbox knew to differentiate with respect to  $x$ , and not  $b$  or  $c$  for example. The answer is that the Symbolic Toolbox uses the variable closest to 'x' alphabetically as its *default variable*.

## 2 Verifying Solutions to Differential Equations

Let's ask MATLAB to find the solution to the logistic model for population growth given by the differential equation

$$\frac{dP}{dt} = k(M - P)P, \quad k = 0.000025, \quad M = 1000 \quad (1)$$

where  $P(0) = 200$ . With  $k = 0.000025$  and  $M = 1000$ , equation 1 becomes  $dP/dt = 0.000025(1000 - P)P$ . Using Matlab's function `dsolve` we can find the solution to the diff-EQ,

```
>> P = dsolve('DP= 0.000025*(1000-P)*P', 'P(0) = 200', 't')
```

we find that the solution of the initial value problem is

$$P = \frac{1000}{1 + 4e^{-t/40}} \quad (2)$$

The notation for  $P'$  is Dy and  $P''$  is D2y, and so forth.

Now, let's plot our diff-EQ

```
>> ezplot(P, [0,300])
>> grid
>> xlabel('Time in minutes')
>> ylabel('Population')
>> title('A Logistic Model')
```

**Problem #1:** Using the graph, approximate the length of time it takes the population to reach a level of 800.

We could solve by hand for the exact value of  $t$  in  $P(t) = 800$ , but instead let's have Matlab do the pencilwork for us. The `solve` command is the Symbolic Toolbox's versatile equation solver.

```
>> s=solve('800=1000/(1 + 4*exp(-t/40))', 't')
>> double(s)
```

Notice that we changed the format of the output by using the `double` command.

**Problem #2:** Exactly, how many minutes does it take the population to reach 800?

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Recall that the logistic population growth equation we used to model the U.S. population growth over the past 200 years is given by

$$\frac{dP}{dt} = k(M - P)P, \quad k = 8.9 \times 10^{-5}, \quad M = 311 \tag{3}$$

where the population level,  $P(t)$ , is expressed in billions and  $t$  is in years, with initial condition,  $P(0) = P_0 = 5.308$ . Let's experiment with the model using `dfield7`.

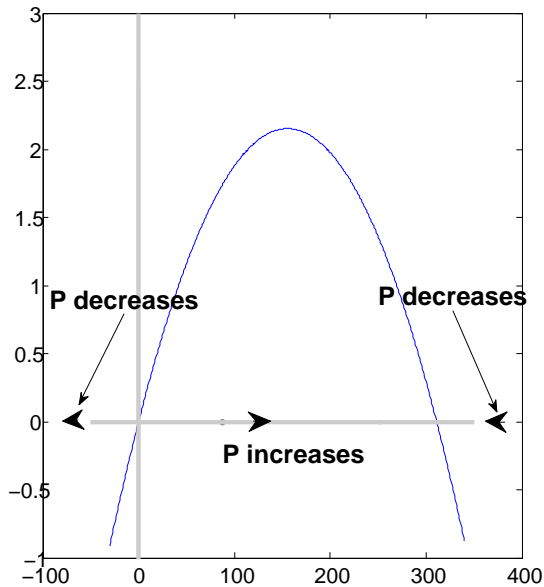


Figure 1: The plot of  $k(M - P)P$  versus  $P$ .

The roots of the inverted parabola are called *equilibrium points*. The solutions  $P(t) = 0$  and  $P(t) = M$  are called *equilibrium solutions*. For example, if  $P(t) = M$ , the growth rate,  $dP/dt$  of the population is zero and the population remains at  $P(t) = M$  forever.

A *phase line* summarizes the slope changes from positive to negative. If the population is greater than  $M$  billion then the population **decreases** to  $M$  billion. For this reason the parameter  $M$  is called the *carrying capacity*.

### Problem #3:

- Plot the differential equation using `dfield7` by entering the differential equation, along with the two parameters,  $k$  and  $M$ , from  $t = 0$  to  $350$  and  $P = 0$  to  $320$ . Draw the solution to the initial value problem using the initial population data, as given in the last lab.
- Add in a couple more solutions by clicking the mouse on the graph of the slope field. Notice that each of these solutions tends to  $311$  as  $t$  increases. Remember that  $M = 311$ . This is not a coincidence, and we will return to this point later.
- Plot the right hand side of the logistic differential equation (3) versus  $P$  for the range of  $-1 \leq P \leq 320$  (*i.e.* plot  $k(M - P) * P$  versus  $P$ ). The result is an inverted parabola, as seen in figure 1.
- Use two data cursors to approximate the roots of the parabola.
- If  $P(t) = 0$ , the growth rate,  $dP/dt = \underline{\hspace{2cm}}$  and the population remains at  $\underline{\hspace{2cm}}$  for all time.

- (f) The information on the phase line indicates that a population beginning between 0 and \_\_\_\_\_ billion has to increase to the equilibrium value of  $M$  billion.
- (g) Let's go back to the logistic equation slope field. Erase the graphed solution curves by selecting **Edit**→**Erase all solutions**. Add the solution trajectories with the initial condition for the U.S. population data,  $P(0) = 5.308$  billion, and  $P(0) = M$  billion and  $P(0) = 0$  billion. Note that the equilibrium solutions are horizontal lines.
- (h) Add a few solution trajectories for initial conditions a little above and below both equilibrium solutions and fill in the blanks with the best answer.
- (i) Solutions beginning a little above or below the equilibrium solution  $P = \underline{\hspace{2cm}}$  tend to move back toward this equilibrium solution with the passage of time. That is why the solution  $P = \underline{\hspace{2cm}}$  is called an *asymptotically stable* equilibrium solution.
- (ii) Solutions beginning a little above or below the equilibrium solution  $P = \underline{\hspace{2cm}}$  tend to move away from this equilibrium solution with the passage of time. That is why the solution  $P = \underline{\hspace{2cm}}$  is called an *unstable* solution.

**Problem #4:** Use `dsolve` to find the solution to the indicated initial value problem. Use `ezplot` to plot the solution over the indicated time interval.

$$yy' + ty = 0, \quad y(1) = 4, \quad [-4, 4] \tag{4}$$

**Problem #5:** Use `dsolve` to find the solution to the indicated second order differential equation. Use `ezplot` to plot the solution on the indicated time interval. Hint: Use the command `simple` to find the simplified form of the provided solution. Use the help document to get more information about the `simple` command.

$$t^2 y'' = (y')^2 \quad y(1) = 3, \quad y'(1) = 2, \quad [0, 64\pi]$$

**Example:** Find the general solution to the first order differential equation

$$\frac{dy}{dt} + y = t e^t.$$

We know how to do this!

```
>> clear all
>> syms y t
>> y = dsolve('Dy + y= t*exp(t)', 't')
or simply,
>> y = dsolve('Dy + y= t*exp(t)')
gives
y =
    -1/4*exp(t) + 1/2*t*exp(t) + exp(-t)*C1
```

Let's substitute a number for the constant  $C1$  and plot the function,  $y$ , on the time interval  $[-1, 3]$ . Enter the following commands:

```
>> C1 = 2
>> y = subs(y)
>> ezplot(y, [-1,3])
```

The `subs` command substitutes the value of  $C1$  into the symbolic expression  $y$ . The `subs` command is actually a pretty versatile command. For example, if we want to find the value of  $y(t = 1)$ , we enter the commands:

```
>> syms t
>> subs(y,t,1)
ans =
    1.4153
```

**Problem #6:** What's the value of the function,  $y$ , from the previous problem when  $t = 2.5$ ?

**Extra Credit:** Suppose we have a population of 100 individuals at time  $t = 0$  and that the population is correctly modeled by the logistic equation. Suppose that at time  $t = 2$  there are 200 individuals in the population, and that the population approaches steady state at a population of 1000. Plot the population over the interval  $[0, 20]$ . What is the population at time  $t = 10$ ?

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**Quit MATLAB** by clicking on the **File** menu in the upper left corner and choosing **Exit**. Please remember to **Log Off** (from the "Start" menu in the lower left of the screen).