Population Dynamics and First-Order ODE's

Objectives:

1. To understand exponential and logistic growth models
2. To gain knowledge about population dynamics models

Case Study: Invasion of the Toads

"Bufo marinus was introduced to northern Queensland in 1935 in an attempt to control the population of a type of beetle that was ravaging the sugar cane crops. The toads ignored the cane beetles, but began ravaging everything else in sight instead. They have immense appetites, breed by the zillions, and secrete poisonous gunk that makes them unpalatable to all but a tiny handful of native Australian animals."

The following problems are based on the Boston University Differential Equations Project. As is often the case in mathematical modeling, there is no "best answer" for this problem. There are, however, "better answers": the correctness of a mathematical model is determined by how well it agrees with the reality of experimental measurements.

Year	Cumulative Area Occupied (km^2)
1939	32,800
1944	55,800
1949	73,600
1954	138,000
1959	202,000
1964	257,000
1969	301,000
1974	584,000

Note that the area of Queensland is 1,728,000 km^2 and the area of Australia is 7,619,000 km^2.

Record the data in a vector in Matlab. Use the plot command to plot the evolution of the toad population throughout the 35-year period.

Let's try to create a model for the population growth that reflects this evolution. A plausible model of the population dynamics is

$$\frac{dP}{dt} = \lambda P, \quad \lambda \sim 0.08 \tag{1}$$

where $P(t)$ is the population level, and t is years, with initial condition, $P(0) = P_0 = 32,800$. The rationale behind the model is that a change in population arises from births and deaths; if b is the typical birth rate, and d the death rate, then one expects $dP/dt = (b - d)P$. When b and d are constants, we set $\lambda = b - d$ and arrive at equation (1). The constant λ is then a difference between the per-capita birth and death rates, and **the term λP reflects the total number of** ($births - deaths$) **per unit time**. This equation thus describes how a population grows if its per-capita birth and death rates are constant.

Problem #1:

(a) Plot the evolution of the toad population. Label the axes and include a title for the graph.

(b) Solve the initial value problem analytically. (*HW #3*)

(c) Compute the predicted population for the year 2039.

(d) Compare the solution to the actual data. Taking into consideration the size of Queensland, do you believe your prediction?

Remark: The American marine toad was introduced to Australia to control sugar cane beetles and in the words of J.W. Hedgpath (see *Science*, July 93 and *The New York Times*, July 6, 1993). "Unfortunately the toads are nocturnal feeders and the beetles are abroad by day, while the toads sleep under rocks, boards and burrows. By night the toads flourish, reproduce phenomenally well and eat up everything they can find. The cane growers were warned by Walter W. Froggart, president of the New South Wales Naturalist Society, that the introduction was not a good idea and that the toads would eat the native ground fauna. He was immediately denounced as an ignorant meddlesome crank. He was also dead right."

Case Study 2: U.S. Population Growth

We are given the following data for the U.S. population from the U.N. Populations Division:

Year	1800	1810	1820	1830	1840	1850	1860	1870
Pop (mil)	5.308	7.240	9.638	12.861	17.064	23.192	31.443	38.558

Year	1880	1890	1900	1910	1920	1930	1940
Pop (mil)	50.189	62.980	76.212	92.228	106.022	123.203	132.165

Year	1950	1960	1970	1980	1990
Pop (mil)	151.326	179.323	203.302	226.542	248.710

Record these data into a vector using Matlab. Let's try to create a model using the exponential growth model from equation 1 to fit the U.S. population growth data over the last two centuries.

Problem #2:

(a) Plot a graph that compares the solution of the exponential model with the actual data. The exponential growth rate coefficient is $\lambda = 0.026643$. Label the axes and include a title for the graph.

(b) Another plausible model of the population dynamics includes an *overcrowding term* and the coefficient of this term is called the *coefficient of overcrowding.* This population model is called the logistic model,

$$\frac{dP}{dt} = k(M - P)P, \quad k = 8.9 \times 10^{-5}, \quad M = 311 \tag{2}$$

where the population level, $P(t)$, is expressed in billions and t is in years, with initial condition, $P(0) = P_0 = 5.308$. The term kM explains the population death rate and they reproduce with a birth rate of kP. The logistic model adds an extra nonlinear term, $-kP^2$, to the exponential model to include the effect that, if the population becomes too large, there is more prevalence for disease, hunger and so on, which implies that the death rate increases with P.

Solve equation (2) analytically to verify that

$$P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{-kMt}}.$$

(c) Plot a graph that compares the solution of the logistic model with the actual data. Label the axes and include a title for the graph.

(d) Which population model better fits the U.S. population growth data?

(e) What is the long-time population ($t \gg 1$)predicted by the exponential model?

(i) and the logistic model?

(f) Do either of the predicted long-time populations (*limiting population*) seem likely? Why or why not?

(g) Which population model better represents the true U.S. population growth? Why?

Problem 3. Suppose that you were modeling the population evolution of a pest using the logistic model, given in equation (2).

(a) Which is more effective in reducing the final size of a pest population: reducing k by 1/2 or reducing M by 1/2? Why?

(b) If you reduce M for a population, how does this change the time for the population to reach M?

Problem 4. Which species population below will recover (*i.e.* reach M) from a catastrophe faster?

(a) Population A with $k = 2$, M = 1000 or Population B with $k = 2$, M = 100?

(b) Population A with $k = 1$, M = 1000 or Population B with $k = 2$, M = 1000?

Quit MATLAB by clicking on the `File` menu in the upper left corner and choosing `Exit`. Please remember to **Log Off** (from the "Start" menu in the lower left of the screen).