Linear Transformations and First-Order ODE's

Objectives:

1. To practice graphing and interpreting phase plots

2. To learn usage of DFIELD display program

1 Solutions for Polynomial Equations

Problem #1 Find a solution to $z^4 + z^2 + 1 = 0$ by considering the equation first as a quadratic for z^2. Matlab has a routine for finding the roots of polynomials, called "roots" (type "help roots"). It operates by feeding in a vector C formed from the coefficients of the polynomial - here $C = [1, 0, 1, 0, 1]$ (since there are no z^3 or z terms). The syntax for the routine is "roots(C)". The roots can be checked by substituting them back into the given equation.

Problem #2 What are the solutions to $z^4 + 2z^3 + z^2 + 2z + 1 = 0$?

2 Slope Fields and Solution Curves

A lot of insight can often be gained about the solutions to ordinary differential equations (ODEs) using graphical methods. For example, consider the first-order equation,

$$\frac{dy}{dx} = -xy, \quad y(0) = y_0 \tag{1}$$

for a constant y_0. We may solve this equation analytically, but first consider the solution in general terms. With the "starting point", $y(0) = y_0$, specified, the solution $y(x)$ defines a curve on the (x, y)−plane. In many problems, the goal is to understand the shape of such "solution curves".

In fact, we can obtain a fair idea of what these curves look like from the ODE itself since the slope of the solution curve, dy/dx, is given by the differential equation itself!

We proceed as follows: consider a host of representative points on the (x, y)-plane, and at each point draw short line segments with the local slope of the solution curves (this the "direction field" or "phase plot" for the ODE). Since the curves must be continuous and smooth, we can then try to sketch out the shapes of the solution curves by threading them through the phase plot in such a way that the curves are tangent to all the line segments they intersect. Finally, we roughly draw the particular solution curve passing through the starting point, $y(0) = y_0$.

NOTE: Open a new figure window for each problem by typing '`figure`' at the Matlab command line, or use the **File** pull-down menu to select **File→New→Figure**.

Problem #3: Let's illustrate the previous explanation with Matlab step by step. First we need a bunch of representative points (X, Y) on the plane:

$$\gg x = [-2.1 : .1 : 2.1]; \ \ y = x; \ \ [X, Y] = \texttt{meshgrid}(\text{x,y});$$

Now let's create the line segments. At each point, the horizontal line segment is given by

```
>>   u=ones(size(X));
```

According to equation 1, the slope at each point with respect to the horizontal line segment is given by

$$\gg v = -X.* \texttt{Y};$$

We will draw the line seqments with the Matlab command "quiver" (**type "help quiver"** in your shell to get a sense of the quiver command).

It is as simple as writing

```
>>   quiver(X,Y,u,v);
```

We will also use a new command: axis. The command axis has to be entered with arguments and should be used after a plot command has been executed. The particular commands used below are "axis tight" and "axis square". These commands resize the plot that you have plotted.

The command "axis tight" resizes the limits of the plot so that the plotted curves fit exactly into the graph (by default, Matlab leaves a little space). The command "axis square" makes the plot come out square (the default is a rectangle). Now, we add some cosmetics to our plot:

```
>>   axis tight, axis square, xlabel x, ylabel y
```

Finally, let's add a few curves showing the exact solution, $y = y_0 e^{-x^2/2}$, for $y_0 = 1$ and $y_0 = -2$:

```
>>   x=[-2.1:.01:2.1], hold on, plot(x,exp(-x.^2/2),x, -2*exp(-x.^2/2)), hold off
```

Now you should see a graph of the direction field for equation 1, as seen in figure (1)
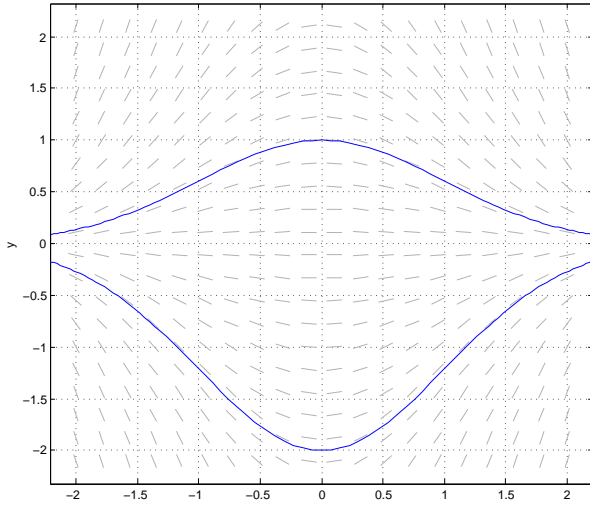
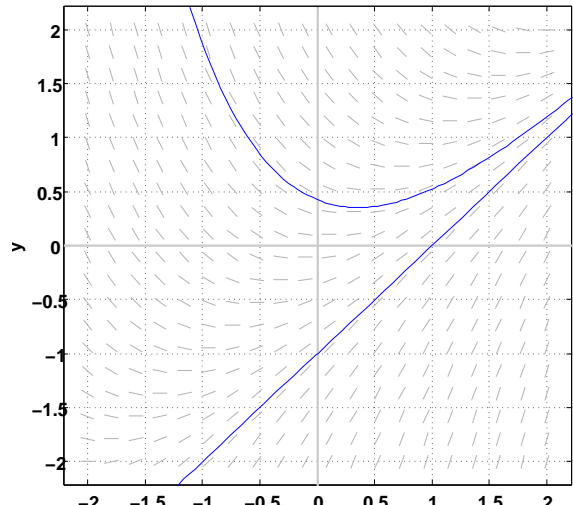Figure 1: Phase plot of equation 1.

Figure 2: Phase plot of equation $y' = x - y$.

If you want to implement the same sequence of commands for a different ODE, all you have to do is insert the right differential equation in the definition of v. The important point is how the little line segments guide the eye in allowing one to roughly sketch out the solution curves.

Alternatively, you can immediately plot a direction field using a free MATLAB programs **DFIELD** and **PPLANE** that are available free for educational use. Our system administrator has already downloaded DFIELD for us, however, for further reference, you can download the free Matlab program at http://math.rice.edu/~dfield/.

Figure 1.3.25 in the text was generated by the MATLAB program **dfield**. When a differential equation is entered in the **dfield** setup menu, you can plot a direction field and then - with a single mouse click - plot also the solution curve through any desired point. For example, figure (2) shows the direction field and typical solution curves for the differential equation $y' = x - y$. It appears that there exists a (single) *straight line* solution curve that all other solution curves approach as $x \to \infty$. Indeed, if we substitute the trial straight line solution $y(x) = ax + b$ in the differential equation, we get

$$a = y' = x - y = x - (ax + b) = (1 - a)x - b$$

which is so, if and only if $a = 1$ and $b = -1$. Thus, $y(x) = x - 1$ is, indeed, a straight line solution of the differential equation $y' = x - y$. The figure then suggests (without proving it) that

$$y(x) - (x - 1) \to 0 \quad \text{as} \quad x \to \infty$$

The next example illustrates several features of **dfield** including how to be accurate with intial conditions.

Problem #4. The voltage y on the capacitor in a certain RC circuit is modeled by the differential equation $y' + y = 3 + \cos x$, where we are using the variable x to represent time. Use `dfield` to plot the voltage over the interval $0 \le x \le 20$, assuming that $y(0) = 1$.

You will notice that we are asked to solve the initial value problem

$$y' + y = 3 + \cos x, \quad y(0) = 1. \tag{2}$$

The dependent variable in this example is y and the independent variable is x. The differential equation $y' + y = 3 + \cos x$ is not is normal form, so we put it in normal form by solving the equation for y', getting $y' = -y + 3 + \cos x$.

Type "`dfield7` <**Enter**>" at the Matlab command prompt. The DFIELD Setup window appears. Select **Edit→Clear all**. Notice that there are options on the Edit menu to clear particular regions of the DFIELD Setup window and each of these options possesses a keyboard accelerator. Enter the left and right sides of the differential equation $y' = -y + 3 + \cos (x)$, the independent variable "x", and define the display window using $0 \le x \le 20$ and $0 \le y \le 4$.

The initial value problem in equation (2) contains no parameters, so leave the parameter fields in the DFIELD Setup window blank. Click the **Proceed** button to transfer the information in the DFIELD Setup window to the DFIELD Display window and start the computation of the direction field.

You can choose the initial point for the solution curve with the mouse, however, it is difficult to be accurate. Instead, in the DFIELD Display window, select **Options→ Keyboard input**. Enter the initial condition, $y(0) = 1$, by assigning the initial value of $x = 0$ and $y = 1$. It isn't necessary to specify a computation interval. Click <**Compute**> to draw the corresponding solution curve.

From the graph of the solution we can see that the voltage $y(18)$ is approximately 3. To get more accuracy, use the **Edit→Zoom in** in the DFIELD Display window, then single-click the (left) mouse button in the DFIELD Display window near the point $(18, 3)$. Another way to zoom in is to click the right mouse button at the zoom point.

Use the cursor position display to find a more accurate value for $y(18)$.

Problem #5: Plot a direction field and typical solution curves for the differential equation $dy/dx = sin(x - y)$, but with a window corresponding to $-10 \le x, y \le 10$. A number of apparent straight line solution curves should be visible.

(a) Substitute $y = ax + b$ (by hand) in the differential equation to determine what the coefficients a and b must be in order to get a solution.

Problem #6: Plot the direction field for the ODE

$$\frac{dy}{dx} = \left(\frac{1}{x} - x\right) y$$

For $y(-1) = y_0$, verify analytically that $y = -y_0\, x \exp\{(1 - x^2)/2\}$. Add the solution curves for $y(-1) = 1$ and $y(-1) = -2$ to your picture. The *limiting behavior* is defined by what happens when $x \to \infty$. What happens eventually to all the solutions?

Problem #7: Consider the following eight first order equations:

$$1.\frac{dy}{dt} = t - 1 \quad 2.\frac{dy}{dt} = t + 1 \quad\quad 3.\frac{dy}{dt} = y + 1 \quad 4.\frac{dy}{dt} = 1 - y$$
$$5.\frac{dy}{dt} = y^2 + y \quad 6.\frac{dy}{dt} = y(y^2 - 1) \quad 7.\frac{dy}{dt} = y - t \quad 8.\frac{dy}{dt} = y + t$$

Four of the associated phase plots are shown in Figure 3. Pair the direction fields with their associated equations – *try it without graphing!* Provide a brief justification for your choice.
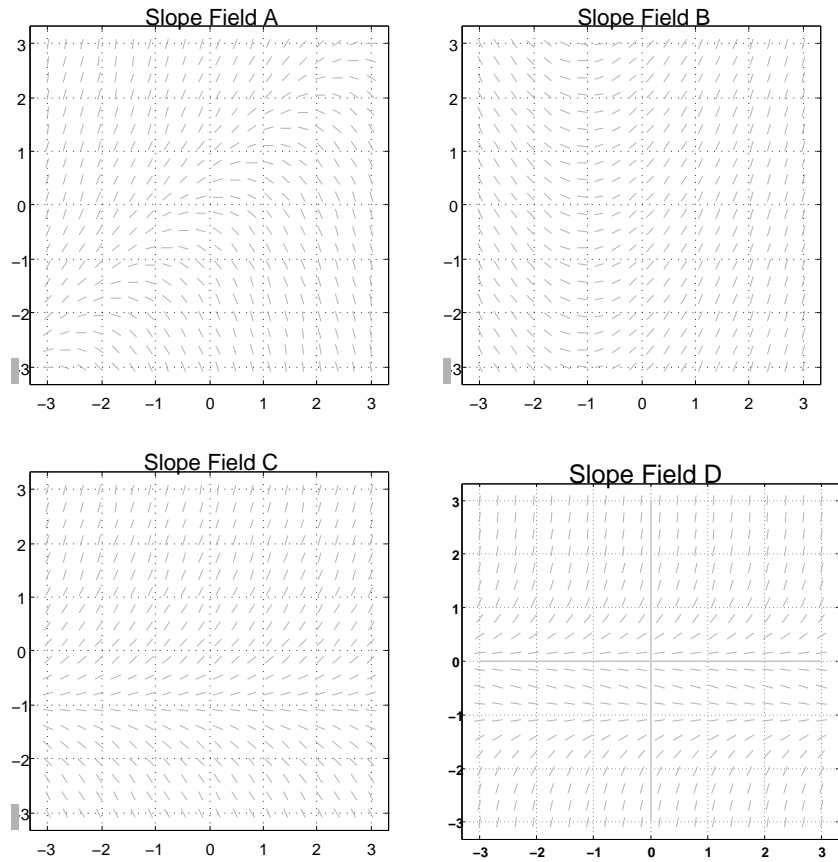
(A)

(B)

(C)

Figure 3: Direction Fields for Problem #6

(D)

Now go back and check your answers by graphing.

Quit MATLAB by clicking on the `File` menu in the upper left corner and choosing `Exit`. Please remember to **Log Off** (from the "Start" menu in the lower left of the screen).